GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Fall Semester 2000

Applied Mathematics
EXAM AREA

Assigned Number **(DO NOT SIGN YOUR NAME)**

- **Please sign your name on the back of this page—**

Applied Mathematics Qualifiers, Fall 2000

Answer 4 out of 5 questions.

1. Find a linearly independent set of two eigenvectors of the matrix

$$\begin{bmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ -2 & -1 & 1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = -2$. Find the other eigenvalue and eigenvector.

to the eigenvector $x_3 = [2 \quad 1 \quad -1]^T$.

2. Find the solution to the first order hyperbolic PDE $f_t + cf_x = 0$, subject to the initial condition $f(x,0) = A(x)$ where c is a constant, by introducing the change of variables $\xi = x - ct$, $\eta = t$.

3. Let **F** = 3x **i** + y **j** + 2z **k**, T is the triangular region that is cut out by the plane x+y+z = 1 and the first octant (x >=0, y>=0, z>=0), **n** is the normal vector of T which points away from the origin.

   (a) Calculate the surface integral

$$\int_T \vec{F} \cdot \vec{n} dA$$

   (b) Let D be the 3-D domain bounded by T, x>=0, y>=0 and z>=0. Evaluate the surface integral over the boundary of D

$$\int_{\partial D} (\nabla \times \vec{F}) \cdot \vec{n} dA$$

   (c) Let **F** be an arbitrary continuously differentiable vector field, D be any finite single-connected domain bounded by at least piecewise smooth surface S. Prove or give a counterexample for the following statement

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n} dA = 0$$

4. The need to calculate the Fourier Transform of experimental data arises in a number of fields. The complex Fourier Transform is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

t = the real "time" variable

ω = the complex frequency variable

f = the real function of time

F = the complex transform of f

$j = \sqrt{-1}$

Experimental data f(t) is assumed to be collected every T seconds beginning at t = 0 and ending at $t = t_f - T$, yielding samples of

$f(iT) = f(t) \quad 0 \le i \le N_t - 1$, i an integer

so that $N_t T = t_f$.

(a) Formulate the calculation of an approximation of F(ω) to be called F*(ω) based on Euler's approximation of integration (the rectangular rule). Formulate your answer in a fashion that could be implemented with real numbers on a computer.

(b) It is also desirable to perform the inverse transformation numerically. The inverse transform is defined as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

where the samples of F(ω) are taken at kΩ, $-(N_f-1) \le k \le (N_f-1)$ are available and Ω is the frequency increment. Formulate this integral also as a numerical integration to approximate f(t) by f*(t). The formulation should be given explicitly in real number operations.

(c) The Fourier transform and the Fourier series are related concepts. What is the fundamental frequency of the Fourier series resulting from the discrete approximation of f(t) by f*(t) in the manner asked for above?

(d) What is the highest frequency component represented in the numerical approximation of the Fourier Transform?

(e) What is the nature of f*(t) for t larger than t_f?

5. The Frobenius method is a very powerful tool for solving ordinary differential equations with variable coefficients

$$x^2\frac{d^2y}{dx^2} + xF(x)\frac{dy}{dx} + G(x)y = 0$$

Here x is the independent variable. The core of this method is an indicial equation.

Suppose that the roots of the indicial equation are different, but not by an integer. The complete solution can be written as

$$y = Au(x, c_1) + Bu(x, c_2)$$

Here c_1 and c_2 are the roots, $u(x, c_1)$ is a series solution to the differential equation.

(1) Explain why the complete solution for the case of two equal roots of the indicial equation is

$$y = \alpha u(x, c_1) + \beta \left(\frac{\partial u}{\partial c}\right)_{c=c_1}$$

Here α and β are constants. You do not need to prove it rigorously. Simply showing the idea is acceptable.

(2) Solve the differential equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} - y = 0$$

You may start with

$$y = \sum_{n=0}^{\infty} a_n x^{n+c}$$