GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Fall Semester 2003

Applied Math
EXAM AREA

Assigned Number (DO NOT SIGN YOUR NAME)

✱ Please sign your **name** on the back of this page —

1. Solve the inhomogeneous ODE given below for $y(x)$ by the method of your choice.

$$y'' - y' - 2y = 10\cos x \qquad \text{for } x \geq 0$$

with $y(0) = 0$

and $y(x)$ is always finite.

2. The problem of solving a linear system of equations is frequently encountered in mechanical engineering. Consider the following non-homogeneous system of linear equations obtained from static equilibrium of known external forces and moments (**F**) and unknown reactions (**c**) acting on a planar rigid object:

$$\mathbf{Wc} = \mathbf{F},$$

where $\mathbf{W} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, and $\mathbf{F} = \begin{bmatrix} 7 \\ -3 \\ -1 \end{bmatrix}$.

a) State the necessary condition(s) for the homogeneous linear system of equations $\mathbf{Wc} = \mathbf{0}$ to have a non-trivial solution.

b) Find the inverse of the matrix **W**. Show all calculations clearly.

c) Solve the linear system of equations $\mathbf{Wc} = \mathbf{F}$ using the Gaussian elimination method.

3. The function $y(x)$ is defined by the following differential equation and initial value:

$$\frac{dy}{dx} = f(x, y) = (2x+3)\,y, \qquad y(0) = 1.$$

Throughout this problem, please perform calculations by hand as far as possible. Show all of your steps.

a) Use Euler's method and a step size of $h = \frac{1}{2}$ to derive an estimate $\hat{y}(1)$, i.e., the estimate of $y(x)$ at $x = 1$. If the exact value is $y(1) = 54.6$, what is the error $\Delta y = y(1) - \hat{y}(1)$?

b) Expand $y(x+h)$ in a second-order Taylor series and use it to derive an improved difference equation of the form:

$$y_{i+1} = y_i + \ldots\ldots\ldots\ldots\ldots$$

c) Use this improved method to come up with a new estimate (using again the step size $h = \frac{1}{2}$) and calculate $\Delta y = y(1) - \hat{y}(1)$.

d) You will find that the error from part (c) is still large. Find the analytical solution for $y(x)$ and explain why the error for this particular problem is so large?

4. Consider two vector fields, **F** and **G**, defined in a three-dimensional domain V enclosed by a surface S with outward unit normal vector **n**. Assume that the following conditions hold:

a) **F** and **G** have the same divergence in V, i.e., $\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \text{div}\,\mathbf{G} = \nabla \cdot \mathbf{G}$ in V.

b) **F** and **G** have the same curl in V, i.e., $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \text{curl}\,\mathbf{G} = \nabla \times \mathbf{G}$ in V.

c) **F** and **G** have the same normal components on S, i.e., $\mathbf{F} \cdot \mathbf{n} = \mathbf{G} \cdot \mathbf{n}$ on S.

Please do the following.

(i) Use $\mathbf{H} = \mathbf{F} - \mathbf{G}$ and Green's theorem to find a relationship between **F** and **G**.

(ii) Explain whether the relationship between **F** and **G** you derived in (i) still holds if the condition (c) is replaced by

c') **F** and **G** have the same boundary value on S, i.e., $\mathbf{F} = \mathbf{G}$ on S.

(iii) Based on your answers to (i) and (ii), make a statement about a vector field.