<u>Instructions</u>

Please complete **4** of the 5 problems attached.

Indicate below which problem to **omit** from grading by striking out the appropriate problem number:

Problem #1

Problem #2

Problem #3

Problem #4

Problem #5

PROBLEM #1

a) With the help of the well known *divergence* theorem:

$$\iiint_V div\, \boldsymbol{q}\; dV = \iint_{\partial V} \boldsymbol{n}\cdot\boldsymbol{q} dA, \qquad (1)$$

[where, using standard notations:

q: arbitrary ("well behaved") vector field, V : volume (spatial region), ∂V : boundary of V (i.e. a *closed* surface $S = \partial V$: volume element, $d\boldsymbol{A} = \boldsymbol{n}dA$: vector element of *area dA*, of *S*,

***n*:** outward unit normal vector, and $div\, \boldsymbol{q} \equiv \nabla\cdot\boldsymbol{q}$: *divergence* of ***q*,**

$\nabla\ldots \equiv \boldsymbol{i}\,\partial\ldots/\partial x + \boldsymbol{j}\,\partial\ldots/\partial y + \boldsymbol{k}\,\partial\ldots/\partial z$: *del/nabla* operator (a symbolic vector), in rectangular Cartesian coordinates/ (ortho-normal-dextral) basis *xyz/**ijk***] show that:

$$\iiint_V curl\;\; \boldsymbol{p}\; dV = \iint_{\partial V} (\boldsymbol{n}\times\boldsymbol{p}) dA \qquad (2)$$

where $curl\;\; \boldsymbol{p} \equiv \nabla\times\boldsymbol{p}$: *curl (-ing)* of arbitrary vector ***p*.**

b) Then, verify that if ***p*** is always normal to a closed surface *S*, then

$$\iiint_V curl\;\; \boldsymbol{p}\; dV = 0 \qquad (3)$$

where *V* is the region bounded by *S*.

Hint: Use the fact that for any three vectors ***a, b, c*** (i.e. identically):

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) \;=\; \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) \;=\; \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) \qquad (4)$$

AND $$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) \;=\; \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b}) \qquad (5a)$$

or: $$div(\boldsymbol{a}\times\boldsymbol{b}) \;=\; \boldsymbol{b}\cdot(curl\; \boldsymbol{a}) - \boldsymbol{a}\cdot(curl\; \boldsymbol{b}) \qquad (5b)$$

PROBLEM #2

(a) Choose either a = 1 or a = 2 and find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

(b) Find, if possible, T such that the eigenvalues of the matrix A appear on the diagonal of $\Lambda = T^{-1}AT$

(c) How would steps (a) and (b) be impacted if you had chosen the other value for variable a? (Note, you do not have to rework the problem unless needed to describe the major impact on both part a and b.)

PROBLEM #3

Consider the equation $m\frac{d^2x(t)}{dt^2}+b\frac{dx(t)}{dt}+kx(t)=A\sin\omega t$ with initial conditions $x\big|_{t=0}=\frac{dx}{dt}\Big|_{t=0}=0$,

where m, b, k, A and ω are constant.

(a) Is this equation (i) an ordinary or partial differential equation? (ii) a linear or non-linear equation? (iii) homogenous or non-homogenous equation?

(b) Given $m=1$, $b=6$, $k=34$, $A=30$ and $\omega=2$, use the Laplace transform method to find the transient and steady state motion for $x(t)$.

(c) Find the magnitude of $x(t\to\infty)$.

$\mathcal{L}[\sin\omega t]=\frac{\omega}{s^2+\omega^2}$ $\quad\mathcal{L}[\cos\omega t]=\frac{s}{s^2+\omega^2}$ $\quad\mathcal{L}\left[e^{-at}\right]=\frac{1}{s+a}$ $\quad\mathcal{L}\left[t^n\right]=\frac{n!}{s^{n+1}}$ $\quad\mathcal{L}\left[e^{-at}f(t)\right]=F(s+a)$

PROBLEM #4

The following boundary value problem deals with waves on a rectangular membrane clamped on two opposite sides (x=0, a) and free on the other two sides (y=0,b).

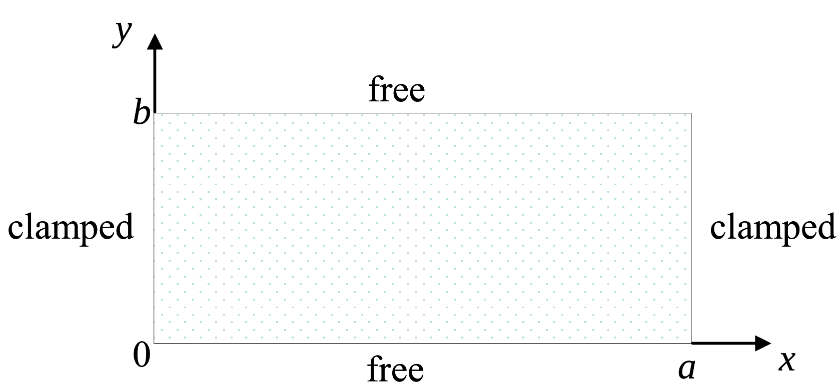
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

where u(x,y,t) is the displacement normal to the surface which is in the x-y plane, t is time, and c is a constant representing the speed of propagation of these waves.

The boundary counditions are:

$u(x=0,y,t) = u(x=a, y, t) = 0$ and $u_y(x, y=0,t) = u_y(x, y=b,t) = 0$,

where $u_y = \partial u / \partial y$. Solve the above PDE and BC by the method of separation of variables.



PROBLEM #5

(a) Use the Newton-Raphson method and four iterations to find x such that

$$f(x) = x^4 - 5 = 0$$

taking $x_0 = 2$ as an initial estimate. (3 points)

(b) Describe the Newton-Raphson method for computing the solution of

$$x + x^2 y^2 - 2y + 3 = 0$$

$$x^3 - 2xy^2 + 2 = 0$$

near (x_0, y_0). (3 points)

(c) Carry out two iterations for the Jacobi method (simple fixed point iteration) for solving

$$\begin{bmatrix} 10 & -1 & -3 \\ 1 & 10 & -2 \\ 3 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

with initial estimate $\underline{x} = \underline{0}$. (3 points)

(d) Describe briefly how the Gauss-Seidel method differs from the Jacobi method. (1 point)