<u>Instructions</u>

Please complete **4** of the 5 problems attached.

Indicate below which problem to **omit** from grading by striking out the appropriate problem number:

Problem #1

Problem #2

Problem #3

Problem #4

Problem #5

Problem 1

Consider the matrix

$$\mathbf{T}=\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Find its principal values (eigen-values), λ_1, λ_2, λ_3 and associated normalized principal directions $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, $\mathbf{n}_1 \cdot \mathbf{n}_1 = 1$, $\mathbf{n}_2 \cdot \mathbf{n}_2 = 1$, $\mathbf{n}_3 \cdot \mathbf{n}_3 = 1$

2. Do the same for the matrix $\mathbf{T}^2$ (i.e., squared), i.e., find Λ_1, Λ_2, Λ_3 and $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$, $\mathbf{N}_1 \cdot \mathbf{N}_1 = 1$, $\mathbf{N}_2 \cdot \mathbf{N}_2 = 1$, $\mathbf{N}_3 \cdot \mathbf{N}_3 = 1$

Problem 2

Consider the vector field $\vec{f}(x,y,z) = x\vec{i} + xz\vec{j} + xy\vec{k}$ in $\|R\|^3$.

a) Determine if the vector field has a potential.

b) Evaluate $\iint_{\Sigma} \vec{f} \cdot d\sigma$, where Σ is the unit sphere $x^2 + y^2 + z^2 = 1$.

c) Evaluate the divergence of the gradient of $\|\vec{f}\|^2$.

d) Consider now the (*x*,*y*) plane in $\|R\|^2$. Evaluate $\int_C (x^2 + y^2)dx + 2xydy$ where C is the path in straight line from (0,0) to (0,2) to (1,2).

Problem 3

$$\frac{d^3x}{dt^3} + (1+p)\frac{d^2x}{dt^2} + (p+q)\frac{dx}{dt} + qx = f(t)$$

(1) Under what conditions is this equation called a linear time-invariant (LTI) homogeneous ordinary differential equation (ODE)? What is the order of this ODE?

(2) What is the characteristic equation? Find the general solution for the homogeneous ODE and discuss the effect of *p* and *q* on the characteristic roots and the form of time response *x(t)*. (Hint: It is already known that e^{-x} is one of the solutions for the homogeneous equation.)

(3) Briefly list the steps about how you will find the solution for the non-homogeneous equation with zero initial conditions.

(4) Under what condition the final value theorem could not be applied to find $x(t \to \infty)$.

Problem 4

(a) Solve the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $u(x,t)$ defined within the domain of $0 \le x \le 1$ and $t > 0$, given the following boundary conditions

(i) $u_x(0,t) = 0$ (note: $u_x \equiv \frac{\partial u}{\partial x}$)

(ii) $u(1, t) = 0$

(iii) $u(x,0) = F(x)$, where $F(x) = 2\cos(0.5\,\pi x) + \cos(3.5\,\pi x)$.

(b) What is the equilibrium solution for the system in (a)? Does your solution in (a) approach the equilibrium solution at large t?

(c) At $x = 1$, temperature is fixed but heat flux is allowed to change. Based on your solution in (a), calculate the heat flux, $\phi \equiv -\frac{\partial u}{\partial x}$ at $x = 1$ as a function of t.

Problem 5

Consider a general problem where $x = f(x)$ and the specific example of this problem $x = -x^3 + 1$

a) Set up the solution for x to be solved using the Regula Falsi also called the False Position method.

b) An alternative numerical method is the Newton-Raphson algorithm. Set up the solution using Newton-Raphson.

c) Step through both algorithms for at least four iterations to complete the table below through $i = 5$

	Regula Falsi			Newton-Raphson		
I	x_i	x_{i+1}	$f(x_{i+1}) - x_{i+1}$	x_i	x_{i+1}	$f(x_{i+1}) - x_{i+1}$
1	0	10		0		
2						
3						
4						
5						
…						

d) Compare the two methods for arbitrary $f(x)$ in terms of

1) Certainty of convergence
2) Applicability
3) Necessary initial information

e) Sketch an example $f(x)$ for which one of the two methods will not converge to a solution.