GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Spring Quarter 1995

APPLIED MATHEMATICS
EXAM AREA

Assigned Number **(DO NOT SIGN YOUR NAME)**

-- Please sign your name on the back of this page --

Qualifying Exam in Applied Mathematics
Spring, 1995

Instructions

This test contains 4 problems; please work all 4 problems completely and be sure to show all your work. Good Luck.

Problem 1.
Recall that a real, symmetric, nxn matrix [A] is *positive definite* if

$$\underline{x}^T [A] \underline{x} > 0 \quad \text{for all } \underline{x} \neq \underline{0}$$

$$\underline{x}^T [A] \underline{x} = 0 \quad \text{only for } \underline{x} = \underline{0}$$

where $\underline{x}$ is a real-valued, nx1 vector.

(a) Show that any symmetric, real matrix with positive eigenvalues ($\lambda_i > 0$ for $i=1,n$) is positive definite. You may assume that the eigenvalues are distinct if necessary.

(b) Consider the matrix:

$$[A] = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

where a is some real-valued scalar. Show that [A] is not positive definite, regardless of the value of a.

Problem 2.
Consider the linear initial value problem:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = f(t)$$

(a) Solve the unforced equation ($f(t)=0$) using the method of Laplace Transforms, subject to the initial conditions $x=0$, $dx/dt=1$ at $t=0$. For your convenience, a table of Laplace Transforms is included in the back of this exam.

(b) Show that the solution to (a) is exactly the response to a unit impulse applied at time $t=0$; i.e., $f(t)=\delta(t)$ and zero initial conditions.

(c) If the same system and i.c.'s of part (a) are now subjected to a specified forcing function $f(t)$, write down an expression for the complete solution in terms of $f(t)$.

Problem 3.

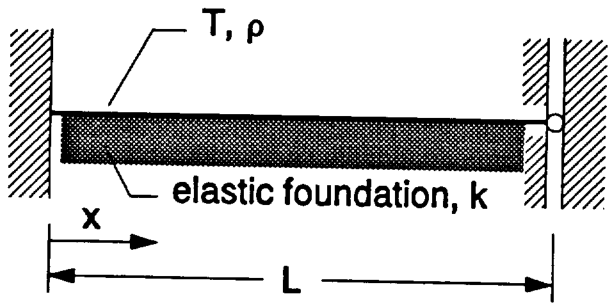
The following partial differential equation corresponds to a fixed-free string on an elastic foundation as shown in the figure below:

$$T\frac{\partial^2 w}{\partial x^2} - kw = \rho\frac{\partial^2 w}{\partial t^2}$$

where T, ρ, and k, are the string tension, string mass per unit length, and foundation stiffness per unit length, respectively, and $w=w(x,t)$ is the transverse string displacement. The associated boundary conditions are $w(0,t)=0$; $\partial w(L,t)/\partial x=0$, where L is the string length. (Assume T, ρ, and L, to be real and positive.)

(a) For the case of $k>0$, find the system eigenvalues, natural frequencies, and eigenfunctions.

(b) Now, consider the case of $k<0$. Explain what happens to the answer found in part (a) as the nondimensional ratio kL^2/T gets more and more negative? Be as specific as you can.



Problem 4.
Evaluate the following line integral using Stokes' Theorem:

$$\oint_C \vec{F}\cdot\vec{ds}\ ; \quad \text{where } \vec{F} = (x+yz)\hat{i} + 2yz\hat{j} + (x-y)\hat{k}$$

and where the curve C is the intersection of the cylinder $x^2+y^2=4$ and the plane $x+y+z=1$.
Recall that Stokes' Theorem is:

$$\oint_C \vec{F}\cdot\vec{ds} = \int_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\,dS$$

(The "hat" in the above expressions designates a unit vector.)

Laplace Transform Pairs and Properties

$f(t)$	$F(s) = \int_0^\infty f(t)e^{-st}dt$
unit impulse $\delta(t)$	1
unit step $1(t)$	$\frac{1}{s}$
ramp t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$a f(t)$	$a F(s)$
$\frac{d}{dt} f(t)$	$sF(s) - f(0)$
$\int_0^t f(t)\,dt$	$\frac{F(s)}{s}$
$\int_0^t f_1(t-\tau) f_2(\tau)\,d\tau$	$F_1(s) F_2(s)$
$e^{-at} f(t)$	$F(s+a)$
$f(t-T)\,1(t-T)$	$e^{-Ts}F(s)$
$t f(t)$	$-\frac{dF(s)}{ds}$

a and T are real-valued scalars; $f(t)$, $f_1(t)$, and $f_2(t)$ are real-valued functions of t