M.E. Ph.D. Qualifier Exam
Spring Semester 2004

GEORGIA INSTITUTE OF TECHNOLOGY

The George W. Woodruff
School of Mechanical Engineering

Ph.D. Qualifiers Exam - Spring Semester 2004

Applied Math
EXAM AREA

Assigned Number (**DO NOT SIGN YOUR NAME**)

* Please sign your **name** on the back of this page —

1. Using forward difference in time and centered difference in space one can approximate the one-dimensional heat diffusion equation,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial^2 x},$$

by the following difference equation,

$$\frac{T_i^{(m+1)} - T_i^{(m)}}{\Delta t} = \kappa \frac{T_{i+1}^{(m)} - 2T_i^{(m)} + T_{i-1}^{(m)}}{(\Delta x)^2}, \quad \begin{matrix} m = 0,1,2,\cdots \\ i = 1,2,\cdots \end{matrix},$$

where

$$T_i^{(m)} = T(t_m, x_i); \quad t_m = m\Delta t; \quad x_i = i\Delta x .$$

(a) Please show that the truncation error of the difference equation is

$$O(\Delta t) + O((\Delta x)^2).$$

(b) Use the difference equation to solve the heat diffusion problem for a bar with a length of 10 cm, a thermal conductivity of $\kappa = 0.835\,\text{cm}^2/s$, and boundary and initial conditions of

$$T(x=0,t) = 100^\circ\text{C}, \quad T(x = 10\,\text{cm}, t) = 50^\circ\text{C}, \quad \text{and} \quad T(t=0) = 0^\circ\text{C}.$$

Use the step sizes $\Delta t = 0.1\,\text{s}$ and $\Delta x = 2\,\text{cm}$.
Perform the calculation until $t = 0.2\,\text{s}$.

(c) Are there any constraints on the step sizes? Why or why not?

2. Eigenvalue problems of the following *general form* are frequently encountered in problems dealing with the vibration of mechanical systems,

$$[A]\mathbf{X} = \lambda[B]\mathbf{X},$$

where [A] and [B] are symmetric matrices of order n, λ is an eigenvalue, and **X** is an eigenvector.

a) If the matrix [B] is known to be diagonal and all of its diagonal elements are positive, develop a procedure to convert the above general form of the eigenvalue problem into its standard form given by $[P]\mathbf{Y} = \lambda\mathbf{Y}$, where the matrix [P] is also symmetric.

b) If $[P] = \begin{bmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{bmatrix}$ and $\mathbf{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$ find the eigenvalues and eigenvectors of this standard eigenvalue problem. Show all calculations.

3. Given the vector field $\underline{F} = x\underline{i} - y\underline{j}$ and the closed surface S composed of the planes $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = a^2$, find the value of the surface integral $\int_S \underline{F} \cdot \underline{n}\, dS$, where $\underline{n}$ is the outward unit normal vector to S. Do this by

a) direct calculation of the integral using cylindrical coordinates, and
b) by using the divergence theorem.

Given the vector field $\underline{F} = -y\underline{i} + x\underline{j}$ and the closed curve C composed of the circle $x^2 + y^2 = a^2$ in the $z = 0$ plane, find the value of the line integral $\int_C \underline{F} \cdot \underline{t}\, ds$, where $\underline{t}$ is the unit tangent vector to C. Do this by

c) direct calculation of the integral using cylindrical coordinates, and
d) by using Stokes' theorem.

4. Integrate the following ODEs and state the solution method used:

a) $(x^2 - y)dx + (y^2 - x)dy = 0$

b) $(3xy - 1)dx + x^2 dy = 0$

c) $2xy^3 dx + (x^2 - 1)dy = 0$.