Special Instructions:

Please complete **4** of the 5 problems attached.

Indicate below which problem to **omit** from grading by striking out the appropriate problem number:

Problem #1

Problem #2

Problem #3

Problem #4

Problem #5

PROBLEM #1

For a given function of $f(x, y) = (x^2 + y^2)e^{-(x^2+y^2)}$,

(a) find all the extreme points where the above function has a local maximum or local minimum;

(b) identify those extreme points giving local maxima and those giving local minima.

PROBLEM #2

Find the solution of (using standard ODE notations):

$$y'' + 3y' = 1 - 9x^2$$

satisfying the conditions:

$$y(0) = 0 \qquad\qquad y'(0) = 1$$

PROBLEM #3

(a) Find the eigenvalues and eigenvectors of the matrix $[\mathbf{Q}] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) Identify any significant properties of [**Q**].

(c) Show that the vector equation $\boldsymbol{a} \times \boldsymbol{x} = \boldsymbol{b}$ can be written in matrix form as

$$skew(\boldsymbol{a})\boldsymbol{x} = \boldsymbol{b}$$

where ***a, b*** are 3-element vectors and $skew(\boldsymbol{a})$ is a skew-symmetric matrix.

(d) Find the rank of $skew(\boldsymbol{a})$ and explain its implication in solving for the unknown vector ***x***.

(e) Determine the eigenvalues of $skew(\boldsymbol{a})$

PROBLEM #4

A string of density (mass per unit length) $\rho(x)$ is stretched along the *x* axis between two points at $x = 0$ and $x = L$. It has tension $T(x)$. The displacement from the *x* axis over time *t* is $y(x, t)$ which is described by the following partial differential equation:

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right] = \rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}, \quad 0 < x < L$$

x is the distance of a point on the string from the left end, a spatial variable, and *t* is time.

(a) Assuming the density ρ is uniform over the distance, and that separation of variables is valid, convert the partial differential equation into (an) ordinary differential equation(s).

(b) Assume further that the tension does not vary significantly with *x*. Based on the result in part (a), determine the nature of the time dependence up to a multiplicative constant(s).

(c) What is the ordinary differential equation that describes the shape of the string at some instant in time?

PROBLEM #5

Consider the following ordinary differential equation, where *t* denotes dimensionless time:

$$\frac{dy}{dt} = -1000y + 3000 - 2000e^{-t}$$

where $y(0)=0$.

While this ODE has an analytical solution, it is instructive to solve it numerically to better understand numerical issues associated with ODE solvers.

Solve by using a numerical method of your choice for $0 \le t \le 0.2$. Discuss the reason behind your choices (e.g., method, step size, etc…)