<u>Instructions</u>

Please complete all **4** problems attached.

Problem 1. Linear Algebra

Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$b = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$$

Without computing A, answer the following questions:

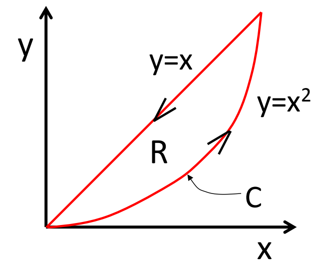
1. Solve $Ax=b$ by solving two triangular systems
2. How do you know that A is symmetric positive definite?
3. Find the determinant of A.

Problem 2. Vector Calculus

Let $R = \{(x, y) \quad | \quad x^2 \le y \le x\}$ and let $\vec{F} = x^3\vec{i} + xy^2\vec{j}$. Compute the integral

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C is a counterclockwise-oriented boundary of R. Is the value of this integral dependent on the specific path of the curve C? Why or why not?



Problem 3. Differential equation

Consider the following equation (defined for $0 < \boldsymbol{x} < \infty$ and $0 \leq \boldsymbol{t} \leq$ T):

$$rf = \frac{\partial f}{\partial t} + rx\,\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2\,\frac{\partial^2 f}{\partial x^2}\,, \quad f = f(t,x)\,,$$

(Eq. 1)

where $\boldsymbol{r}$ and $\boldsymbol{\sigma}$ are constants.

1. Using the following variable substitutions, $\boldsymbol{u=f\exp(-rt)}$, $\boldsymbol{\tau=(T-t)}$, $\boldsymbol{y=\ln(x)}$, and $\boldsymbol{z=y+(r-\sigma^2/2)\tau}$, transform the problem to one of simple and well-known equations of mathematical physics for $\boldsymbol{u(\tau,z)}$.

2. Define the type of the transformed equation and based on your classification, indicate whether the original equation (Eq. 1) needs an initial or final (terminal) condition for $\boldsymbol{f(t,x)}$ to be mathematically well-posed?

3. If this condition (part 2) is specified as a known function $\boldsymbol{\Psi(x)}$, express an analytical solution to the problem $\boldsymbol{f(x,t)}$ in terms of $\boldsymbol{\Psi(x)}$ and the relevant fundamental solution.

Problem 4. **Numerical Analysis**

Evaluate $I = \int_0^2 r^3 dr$ by

(a) the trapezoidal rule using four intervals.

(b) the Simpson's 1/3 rule using four intervals. What is the error reduction if the interval size is reduced by half? Comment on the size of the error in terms of the derivation of Simpson's rule.

(c) What is the order of error reduction in the trapezoidal rule if the interval size is reduced by half? What is the order of error reduction in the Simpson's 1/3 rule if the interval size is reduced by half?

(d) Determine the number of intervals *n* to guarantee an error less than 0.0001 for the trapezoidal rule.